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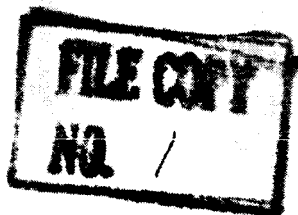
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BY

S. V. VOLLENDER



(Russian text and English translation)

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EQUATIONS OF MOTION OF A VISCOUS GAS

By S. V. Vollender

From Prikladnaya Matematika i Mekhanika. Vol XV, No. 4, 1947

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The present work is devoted to the derivation of the differential equations of motion of a viscous gas. It is assumed here that the gas, to a sufficient degree of accuracy, may be considered an ideal gas; that is, strongly compressed motions of gas are excluded from the reasoning and the ratio of the mean free path of a molecule to a characteristic dimension is small in comparison to unity, that is, motions of extremely rarified gases are excluded.

It is assumed, moreover, that to a sufficiently accurate degree the gas motion to be considered satisfies the law of uniform distribution of internal energy with respect to the degree of freedom of molecule motion, that is, excluded from consideration are cases of gas motion with extremely rapid variation of hydrodynamic elements in space and time.

The regularity of the macroscopic motion of a real gas is connected to the randomness of the microscopic motion since the macroscopic motion results from the microscopic motion of an enormous number of molecules. From the truth (authenticity) of these test data it follows that a finite number of random microscopic motions is so large that the macroscopic motion can be formulated by means of the same statistical laws which we would have in the case that the number of microscopic motions were infinitely large.

Thus instead of a real discrete gas with a finite number of finite molecules, this permits the consideration of a gas "limited" in the form of a continuous continuum of infinitely small molecules.

Such an arrangement is the approximate scheme, with which the regularity of the macroscopic gas motion is conserved, the possibility is overlooked of studying the viscous nature of small fluctuations connected with the finite number of molecules of a real gas.

By replacing the study of a gas by the study of the motion of a material continuum, it is possible to introduce the concepts of density, velocity, complete internal energy of a unit mass of gas. Let us note that by the introduction into the reasoning of a gaseous continuum and the definition of basic hydrodynamic elements, we nowhere create impediments to the consideration of the microscopic motions of molecules and we even, conversely, assume their existence. The introduced gaseous continuum may be allotted all the physical properties of a real gas not related to a finite number of its molecules. In particular it may be required that the mean free path of the particle of the gaseous continuum be, as in a real gas, a finite although always small quantity.

The reproduced below derivation of the equations of motion of a viscous gas shows that the usually applied [1,2,3,4] equations are obtained from insufficiently complete physical representations. Consequently a series of terms of the same order of smallness, absent in them, are kept.

#### 1. Fundamental Concepts

Let us denote by  $m$  the mass of gas in some volume. Then the density  $\rho$  of the gas at a given point  $M$  at a given time  $t$  will be

called the limit of the ratio of the mass  $m$  at time  $t$  to the volume  $P$ , if the latter which envelops the point  $M$  is contracted to this point.

Since the momentum of a system equals the momentum of the center of inertia of a system in which all the mass of the system is concentrated, then the introduced velocity  $\vec{v}$  of the gas motion is the velocity of the center of inertia of an infinitely small volume.

Let us introduce the concept of total energy  $u$  and interior energy  $E$  of a unit mass of gas.

Let us consider again some volume  $P$  enclosing the point  $M$ . Inside the volume  $P$  will be found some mass of gas possessing some finite energy, because by representing the gas ideally, then all the energy of these molecules will be kinetic. Let us denote it by  $U_1$ . Then the condition to express the total energy  $U$  of a unit mass of gas at a given point in a given moment of time  $t$  is the limit of the ratio of  $U_1$  at  $t$  to a mass of gas  $m$  at time  $t$  if the volume  $P$  enclosing  $M$  shrinks to this point.

On the basis of Koenig's theorem the quantity  $U_1$  may be divided into two components: the kinetic energy  $U_2$  of the center of inertia and the kinetic energy  $U_3$  of the relative motion with respect to the center of inertia.

The internal energy  $E$  of unit mass of gas at a given point  $M$  at time  $t$  we will express by the limit of the ratio of the kinetic energy  $U_3$  (at time  $t$ ) of the relative motion of molecules with respect to the center of inertia to the mass of gas  $m$  (at time  $t$ ) if  $P$  enclosing  $M$ , shrinks to the point.

If the mass, momentum, total and internal energy of the gas included in the element of volume  $dP$  be denoted by  $dm$ ,  $\vec{dK}$ ,  $dU^*$  and  $dE^*$ , then from the definitions introduced it follow immediately

$$\begin{aligned} dm &= \rho dP \\ \vec{dK} &= \rho \vec{v} dP \\ dU^* &= \left( \frac{1}{2} \rho v^2 + \rho E \right) dP \\ dE^* &= \rho E dP \end{aligned} \tag{1.1}$$

The quantity  $E$  of internal energy of a unit mass of gas which is introduced above may evidently be divided into two components.

The first of these  $E_1$  will correspond to that part of the kinetic energy of the molecular motion which is related to its successive motions.

The second of these  $E_2$  will correspond to that part of the kinetic energy of the molecular motion which is a result of its rotatory and oscillatory motion.

The quantity, proportioned to  $E_1$ , in the kinetic theory of gases is the temperature. The proportionality factor evidently depends on the units in which temperature is measured and becomes completely determined if such a unit is chosen. Usually temperature is measured in degrees Kelvin and is denoted by  $T$ .

For these units of measurement defining temperature we have

$$m_0 E_1 = 3/2 kT, \quad k = 1.37 \times 10^{-16} \text{ erg/deg.} \tag{1.2}$$

where  $k$  is the so-called Boltzmann constant and  $m_0$  is the mass of a gas molecule.

Since the law of uniform distribution of energy by free degrees occurs for the considered gas motion, then the part  $E_2$  of  $E$  is

proportional to  $T$ .

Therefore

$$E = c_v T \quad (1.3)$$

where  $c_v$  is a new proportionality coefficient called the specific heat for constant volume.

### Transfer Occurrences and Some Conclusions

Let us give an elementary treatment of some physical occurrences belonging to the so-called transfer occurrences.

1. Density of self-diffusion: Let us consider some fixed area  $A B C D$  of the area  $\Delta S$  with normal  $\vec{n}$  in a macroscopic gas. Let us denote by  $\lambda$  the molecular mean free path and by  $c$  the mean value of the velocity of the thermal molecular motion and assume that the gas temperature is constant.

Let us simplify the representation of the molecular motion and consider that half the molecules has a velocity in the normal direction  $\vec{n}$  and half in the opposite direction.

Moreover, let us consider that all the molecules traverse the path  $\lambda$  without collision in the time

$$\Delta t = \frac{\lambda}{c} \quad (2.1)$$

Then half the mass of the gas layer of thickness  $\lambda$  above, the area  $\Delta S$  go down and half the mass of the gas layer of thickness  $\lambda$ , below  $\Delta S$ , go above.

If  $\lambda$  is small in comparison with the characteristic dimension of the occurrence  $l$ , then, with sufficient accuracy, we may write

$$\Delta m_2 = \Delta S \lambda \left( \rho + \frac{\partial \rho}{\partial n} \frac{\lambda}{2} \right), \quad \Delta m_1 = \Delta S \lambda \left( \rho - \frac{\partial \rho}{\partial n} \frac{\lambda}{2} \right) \quad (2.2)$$

where  $\Delta m_2$  is the mass of a gas layer of thickness  $\lambda$  situated above the area  $\Delta S$ ,  $\Delta m_1$  is the mass of a gas layer of thickness  $\lambda$  lying below the area  $\Delta S$ ,  $\rho$  is the gas density at some point of the area,  $\partial \rho / \partial n$  is the derivative in the  $\vec{n}$  direction of the density  $\rho$  at some point of  $\Delta S$ .

Evidently, then

$$\Delta m = \frac{1}{2} (\Delta m_2 - \Delta m_1) = \frac{1}{2} \Delta S \lambda \frac{\partial \rho}{\partial n} \quad (2.3)$$

yields the mass of the gas transported as a consequence of the inconstancy of the density across the element of area  $\Delta S$  in time  $\Delta t$ .

Let us denote by  $Q_{np}$  the mass flow across the area with normal  $\vec{n}$  in the direction opposite to  $\vec{n}$  connected with the inconstancy of the density.

Then

$$Q_{np} = \frac{\Delta m}{\Delta S \Delta t} = \frac{1}{2} \lambda c \frac{\partial \rho}{\partial n} \quad (2.4)$$

Because of the simplicity of the representation of the motion of the gas molecule, it is impossible to guarantee the correctness of the numerical factor in (2.4). Consequently, put

$$Q_{np} = f_1 \lambda c \frac{\partial \rho}{\partial n} \quad (2.5)$$

where  $f_1$  is the nondimensional numerical factor of the order of unity.

From (2.5) we see that with the variation of density in the macroscopic quiescent gas occurs the flow of mass across an area immobile with respect to the gas. This phenomenon of mass flow because of the variation of density it is expedient to call the density of self-diffusion.

Evidently this reasoning is preserved in the case when the macroscopic motion of gas and area are considered moving in space with velocity  $\vec{v}$  of the macroscopic gas motion. The self-diffusion of the density the mass flow  $Q_{np}$  in this case will also be given by (2.5).

Let us observe that the density of self-diffusion is never taken into account in describing the motion of a viscous gas.

2. Density of Heat Conduction: Let us consider, as above, certain fixed elements of area  $\Delta S$  with normal  $\vec{n}$  in the macroscopic quiet gas, let us retain the previous notation and let us consider the question of the transport of the internal energy across the area  $\Delta S$  assuming the gas temperature constant.

Downward will be transported mass  $1/2 \Delta m_2$  with internal energy  $\Delta E_2$ , upward will be transported the mass  $1/2 \Delta m_1$  with internal energy  $\Delta E_1$ . Evidently

$$\Delta E_2 = \frac{1}{2} \Delta m_2 c_v T, \quad \Delta E_1 = \frac{1}{2} \Delta m_1 c_v T \quad (2.6)$$

The quantity

$$\Delta E = \Delta E_2 - \Delta E_1 \quad (2.7)$$

gives the amount of internal energy transported because of the inconstancy of the density across the area  $\Delta S$  during time  $\Delta t$ .

Let us denote by  $t_{np}$  the flow of the internal energy across an area with normal  $\vec{n}$  in a direction  $\vec{n}$  opposite to  $\vec{n}$  connected with the inconstancy of the density. We have

$$t_{np} = \frac{\Delta E}{\Delta S \Delta t} = f_1 \lambda c_v T \frac{\partial \rho}{\partial n} \quad (2.8)$$

if the correctness of the numerical factor is not certified.



then put

$$t_{np} = f_2 \kappa c_v T \frac{\partial \rho}{\partial n} \quad (2.9)$$

where  $f_2$  is a numerical factor of the order of unity.

It is evident that the reasoning fails in the case when the macroscopic motion of the gas and the area  $\Delta S$  are considered moving with the velocity  $v$  of the macroscopic gas motion. Formula (2.9) will yield in this case the flow of internal energy related to the variability of the density.

The phenomenon of energy transport across an area moving along with the gas which arises from the variability of the density, is called the density of heat conduction.

Let us note that the phenomenon of the density of heat conduction is never taken into account in the derivation of the equations of motion of a viscous gas.

3. Viscosity: Using the same simplification of the scheme of molecular motion, it is not difficult to consider the question of momentum transfer across an area moving with the gas for those cases of motion when the macroscopic velocity of the gas motion varies in space.

The coefficients of viscosity  $\mu_1$  and  $\mu$ , which appear here, consist of the products of the velocity components with respect to the coordinates in expressions for momentum flow and are obtained from the formulas

$$\mu_1 = f_3 \rho \lambda c \quad \mu = f_4 \rho \lambda c \quad (2.10)$$

where  $f_3$  and  $f_4$  are numerical factors of the order of unity.

Note. The transport phenomenon is not exhausted by the three considered phenomena if only because the average of the considered phenomena is not

the usual temperature of heat conduction.

Nevertheless it is possible to make three essential conclusions from consideration of these three phenomena.

First: the usual reasoning may not be used for the components of the equations of motion of a viscous gas with a fluid volume of constant mass because it may vary at the expense of the self-diffusion of the mass flows of a mass volume bounded by a closed surface moving with the gas.

Second: it is impossible to be limited only to the consideration of viscosity and the usual temperature of heat conduction for the components of the equations of motion of a viscous gas because there exist other transport phenomena.

Third: it should be kept in mind for the components of the equations of motion of a viscous gas that mass, momentum and energy are transported across a surface moving with the gas for which the hydrodynamic elements vary in space.

### 3. General Description of the Laws of Variation

Let us consider the average of some fixed volume  $V$  bounded by a surface  $S$  moving with a velocity  $\vec{v}$  in space and let us assume that some scalar or vector quantity  $A$ , a function of the coordinates and time, is defined at points of the moving medium which fills the space.

Let us also consider together with  $A$  the quantity  $\Phi$  defined by the formula

$$\Phi = \iiint_V A \, dV \quad (3.1)$$

where  $dV$  is an element of the volume  $V$ .

For a fixed volume  $V$ , the quantity  $\Phi$  will be a function only

of time  $t$  and evidently

$$\frac{d\Phi}{dt} = \iiint_V \frac{\partial A}{\partial t} dV \quad (3.2)$$

Let us admit that the variation of  $\Phi$  with time results only from consideration of the independent action of the following two factors:

1) Inside the volume  $V$  the quantity  $\Phi$  with volume velocity  $B$  results at the expense of the effect of this factor in the volume  $dV$  in time  $dt$  so that  $\Phi$  undergoes a variation determined by the formula

$$\Delta_1 \Phi = B dV dt \quad (3.3)$$

2) The flow of  $\Phi$  across the surface  $S$  of the volume  $V$  with surface density  $G_n$  occurs since because of the effect of this factor on the surface element  $dS$  with external normal  $\vec{n}$  in time  $dt$  the quantity  $\Phi$  undergoes a variation  $\Delta_2 \Phi$  defined by the formula

$$\Delta_2 \Phi = G_n dS dt \quad (3.4)$$

Since both factors act independently of each other, then by integrating  $\Delta_1 \Phi$  over  $V$  and  $\Delta_2 \Phi$  over  $S$ , combining the results of the integrations and dividing by  $dt$ , we obtain the second expression for  $d\Phi/dt$ , given by the formula

$$\frac{d\Phi}{dt} = \iiint_V B dV + \iint_S G_n dS \quad (3.5)$$

Equating (3.2) and (3.5), we arrive at the equation

$$\iiint_V \frac{\partial A}{\partial t} dV = \iiint_V B dV + \iint_S G_n dS \quad (3.6)$$

Let us carry out all the considerations in the arbitrary, orthogonal curvilinear coordinates  $q_1, q_2, q_3$  which are related to Cartesian coordinates by dependencies not containing time and let us choose from the

volume  $V$  bounded by the surfaces

$$q_1 = a, \quad q_1 = q_1; \quad q_2 = b, \quad q_2 = q_2; \quad q_3 = c, \quad q_3 = q_3 \quad (3.7)$$

If the Lamé coefficients be denoted by  $H_1, H_2$ , and  $H_3$  then (3.6)

is written for the chosen volume in the following way:

$$\begin{aligned} \int_a^{q_1} \int_b^{q_2} \int_c^{q_3} \frac{\partial}{\partial t} H_1 H_2 H_3 dq_1 dq_2 dq_3 &= \int_a^{q_1} \int_b^{q_2} \int_c^{q_3} B H_1 H_2 H_3 dq_1 dq_2 dq_3 + \\ &\int_b^{q_2} \int_c^{q_3} \left[ G_1(q_1, q_2, q_3, t) H_2(q_1, q_2, q_3) H_3(q_1, q_2, q_3) + \right. \\ &\quad \left. G_{-1}(a, q_2, q_3, t) H_2(a, q_2, q_3) H_3(a, q_2, q_3) \right] dq_2 dq_3 + \\ &\int_c^{q_3} \int_a^{q_1} \left[ G_2(q_1, q_2, q_3, t) H_3(q_1, q_2, q_3) H_1(q_1, q_2, q_3) + \right. \\ &\quad \left. G_{-2}(q_1, b, q_3, t) H_3(q_1, b, q_3) H_1(q_1, b, q_3) \right] dq_3 dq_1 + \\ &\int_a^{q_1} \int_c^{q_3} \left[ G_3(q_1, q_2, q_3, t) H_1(q_1, q_2, q_3) H_2(q_1, q_2, q_3) + \right. \\ &\quad \left. G_{-3}(q_1, q_2, c, t) H_1(q_1, q_2, c) H_2(q_1, q_2, c) \right] dq_1 dq_2 \end{aligned} \quad (3.8)$$

where  $G_1, G_2, G_3$  denote the surface density of the stream across the boundaries of a curvilinear parallelepiped with normals parallel to the  $q_1, q_2, q_3$  axes and  $G_{-1}, G_{-2}, G_{-3}$  denote the same quantities for the oppositely directed normal.

Differentiating both sides of (3.8) with respect to  $q_1, q_2, q_3$  easily yields

$$\frac{\partial}{\partial t} = B + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial(G_1 H_2 H_3)}{\partial q_1} + \frac{\partial(G_2 H_3 H_1)}{\partial q_2} + \frac{\partial(G_3 H_1 H_2)}{\partial q_3} \right] \quad (3.9)$$

Let us separate into two each of the quantities  $G_1, G_2, G_3$

putting

$$G_1 = -v_1 A + C_1 ; \quad G_2 = -v_2 A + C_2 ; \quad G_3 = -v_3 A + C_3 \quad (3.10)$$

where  $v_1, v_2, v_3$  are the projections of the velocity vector  $\vec{v}$  of the medium on the  $q_1, q_2, q_3$  axes.

The sense of the components in formula (3.10) are completely clear. If the moving medium be displaced only as the ordinary (not gasiform) deformable medium and if the flow of the quantity  $\Phi$  is related only to the macroscopic motion of the substance across the surface  $S$ , then we will only have the first components in formula (3.10). In reality, because of the molecular structure of the real medium, the flow across the surface may be related not only to the macroscopic motion, but to the molecular motions inside the substance moving with velocity  $\vec{v}$ . Consequently corrections, which are denoted by  $C_1, C_2, C_3$ , to the first components are necessary. These corrections are only the flows across surfaces moving with the velocity of the medium.

Putting (3.10) in (3.9) we obtain

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial (A v_1 H_2 H_3)}{\partial q_1} + \frac{\partial (A v_2 H_3 H_1)}{\partial q_2} + \frac{\partial (A v_3 H_1 H_2)}{\partial q_3} \right] = \\ B + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial (C_1 H_2 H_3)}{\partial q_1} + \frac{\partial (C_2 H_3 H_1)}{\partial q_2} + \frac{\partial (C_3 H_1 H_2)}{\partial q_3} \right] \quad (3.11) \end{aligned}$$

or

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{v_1}{H_1} \frac{\partial A}{\partial q_1} + \frac{v_2}{H_2} \frac{\partial A}{\partial q_2} + \frac{v_3}{H_3} \frac{\partial A}{\partial q_3} + \\ \frac{A}{H_1 H_2 H_3} \left[ \frac{\partial (v_1 H_2 H_3)}{\partial q_1} + \frac{\partial (v_2 H_3 H_1)}{\partial q_2} + \frac{\partial (v_3 H_1 H_2)}{\partial q_3} \right] = \\ B + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial (C_1 H_2 H_3)}{\partial q_1} + \frac{\partial (C_2 H_3 H_1)}{\partial q_2} + \frac{\partial (C_3 H_1 H_2)}{\partial q_3} \right] \quad (3.12) \end{aligned}$$

If we take advantage of the well-known formulas

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial t} + \frac{v_1}{H_1} \frac{\partial A}{\partial q_1} + \frac{v_2}{H_2} \frac{\partial A}{\partial q_2} + \frac{v_3}{H_3} \frac{\partial A}{\partial q_3} \\ \text{div } \vec{v} &= \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (v_1 H_2 H_3) + \frac{\partial}{\partial q_2} (v_2 H_3 H_1) + \frac{\partial}{\partial q_3} (v_3 H_1 H_2) \right] \end{aligned} \quad (3.13)$$

then equation (3.12) can be written in the following form<sup>1</sup>:

$$\frac{dA}{dt} + A \text{ div } \vec{v} = B + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (C_1 H_2 H_3) + \frac{\partial}{\partial q_2} (C_2 H_3 H_1) + \frac{\partial}{\partial q_3} (C_3 H_1 H_2) \right] \quad (3.14)$$

This equation represents the desired record, in differential form, of the general law of variation of  $A$  with the assumptions made before on the factors defining the variation of  $\Phi$  which is related to  $A$  by (3.1).

#### 4. Equations of Motion of a Medium

The equations of motion of a medium taking account of mass flow, momentum, and energy are derived very simply with the aid of (3.14), if the laws of conservation of mass, momentum and energy are interpreted by the laws of variation as expressed by (3.14).

To obtain the continuity equation, it is necessary to take the mass conservation law and, assuming the absence of a three dimensional distribution of sources, to put

$$\Phi = M, \quad A = \rho, \quad B = 0, \quad C_1 = Q_1, \quad C_2 = Q_2, \quad C_3 = Q_3 \quad (4.1)$$

1. Equation (3.14) may be obtained by considering a moving volume  $V$  bounded by a surface  $S$  the points of which are moving with the macroscopic velocity  $\vec{v}$  of the motion of the medium.

In this case there is obtained instead of (3.6)

$$\iiint_V \left( \frac{\partial A}{\partial t} + \text{div } A \vec{v} \right) dV = \iiint_V B dV + \iint_S C_n dS \quad (3.6a)$$

where  $C_n$  is the flow of  $\Phi$  across an area moving with the gas and having a normal  $\vec{n}$ . The sense of the quantities  $C_1, C_2, C_3$  is disclosed simultaneously from (3.6a), (3.14) and by inspection.

where  $M$  is the mass,  $\rho$  is the density,  $Q_1, Q_2, Q_3$  are the mass flows as a consequence of self-diffusion across an area perpendicular to the coordinate axes.

Then this equation is obtained

$$\frac{d\rho}{dt} + \rho \operatorname{div} \vec{v} = \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_3 H_1) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] \quad (4.2)$$

To obtain correctly the equations of motion, it is necessary to take the momentum law and put

$$\Phi = \vec{K}, \quad A = \rho \vec{v}, \quad B = \rho \vec{F}, \quad C_1 = \vec{\tau}_1, \quad C_2 = \vec{\tau}_2, \quad C_3 = \vec{\tau}_3 \quad (4.3)$$

where  $\vec{K}$  is the momentum,  $\vec{v}$  is velocity,  $\vec{F}$  is the mass force  $\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3$  are the momentum flows across a surface perpendicular to the axes or what is the same, the pressure of the surface forces.

Putting (4.3) into (3.14) we obtain

$$\frac{d}{dt} (\rho \vec{v}) + \rho \vec{v} \operatorname{div} \vec{v} = \rho \vec{F} + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (\vec{\tau}_1 H_2 H_3) + \frac{\partial}{\partial q_2} (\vec{\tau}_2 H_3 H_1) + \frac{\partial}{\partial q_3} (\vec{\tau}_3 H_1 H_2) \right] \quad (4.4)$$

or

$$\rho \frac{d\vec{v}}{dt} + \vec{v} \left( \frac{d\rho}{dt} + \rho \operatorname{div} \vec{v} \right) = \rho \vec{F} + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (\vec{\tau}_1 H_2 H_3) + \frac{\partial}{\partial q_2} (\vec{\tau}_2 H_3 H_1) + \frac{\partial}{\partial q_3} (\vec{\tau}_3 H_1 H_2) \right] \quad (4.5)$$

Replacing the brackets in the left side of (4.5) with the aid of (4.2)

we obtain

$$\rho \frac{d\vec{v}}{dt} + \frac{\vec{v}}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_3 H_1) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] = \rho \vec{F} + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (\vec{\tau}_1 H_2 H_3) + \frac{\partial}{\partial q_2} (\vec{\tau}_2 H_3 H_1) + \frac{\partial}{\partial q_3} (\vec{\tau}_3 H_1 H_2) \right] \quad (4.6)$$

or

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$$\begin{aligned} \rho \frac{d\vec{v}}{dt} + \frac{\vec{v}}{H_1 H_2 H_3} \left[ \frac{\partial (Q_1 H_2 H_3)}{\partial q_1} + \frac{\partial (Q_2 H_3 H_1)}{\partial q_2} + \frac{\partial (Q_3 H_1 H_2)}{\partial q_3} \right] = \\ \rho \vec{F} + \frac{1}{H_1} \frac{\partial \vec{\tau}_1}{\partial q_1} + \frac{1}{H_2} \frac{\partial \vec{\tau}_2}{\partial q_2} + \frac{1}{H_3} \frac{\partial \vec{\tau}_3}{\partial q_3} + \frac{\vec{\tau}_1}{H_1 H_2 H_3} \frac{\partial (H_2 H_3)}{\partial q_1} + \\ \frac{\vec{\tau}_2}{H_1 H_2 H_3} \frac{\partial (H_3 H_1)}{\partial q_2} + \frac{\vec{\tau}_3}{H_1 H_2 H_3} \frac{\partial (H_1 H_2)}{\partial q_3} \end{aligned} \quad (4.7)$$

In practice, equation (4.7) is not used in its vector form, but in projections on the curvilinear coordinate axes. Consequently, it is necessary to project the vectors



entering in this equation on the curvilinear coordinate axes.

The formulas for the projection of the acceleration  $\vec{dv}/dt$  are known:

$$\begin{aligned} H_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{v_1}{H_1} \frac{\partial v_i}{\partial q_1} + \frac{v_2}{H_2} \frac{\partial v_i}{\partial q_2} + \frac{v_3}{H_3} \frac{\partial v_i}{\partial q_3} + \frac{v_1 v_{i+1}}{H_1 H_{i+1}} \frac{\partial H_i}{\partial q_{i+1}} + \\ \frac{v_1 v_{i+2}}{H_1 H_{i+2}} \frac{\partial H_i}{\partial q_{i+2}} - \frac{v_{i+1}^2 \partial H_{i+1}}{H_i H_{i+1}} \frac{\partial}{\partial q_i} - \frac{v_{i+2}^2 \partial H_{i+2}}{H_i H_{i+2}} \frac{\partial}{\partial q_i} \quad (i = 1, 2, 3) \end{aligned} \quad (4.8)$$

where  $v_i = v_{i+3}$  is the projection of the velocity on the curvilinear coordinate axes and  $H_i = H_{i+3}$  are Lamé coefficients.

To obtain formulas for the projections of the derivatives of the vectors  $\vec{r}_1, \vec{r}_2$  and  $\vec{r}_3$  with respect to the coordinates is also represented without difficulty. But

$$\frac{\partial \vec{r}_k}{\partial q_l} = B_{kl}^{(1)} \vec{i}_1 + B_{kl}^{(2)} \vec{i}_2 + B_{kl}^{(3)} \vec{i}_3 \quad (k = 1, 2, 3; l = 1, 2, 3) \quad (4.9)$$

where  $\vec{i}_1, \vec{i}_2, \vec{i}_3$  are orthogonal curvilinear coordinates and the  $B_{kl}^{(m)}$  are coefficients which are found later.

Taking into account (4.9), then for the derivative of any vector  $\vec{a}$  with respect to the coordinates  $q_l$

$$\frac{\partial \vec{a}}{\partial q_l} = \sum_{j=1}^3 \left( \frac{\partial a_j}{\partial q_l} + a_1 B_{1l}^{(j)} + a_2 B_{2l}^{(j)} + a_3 B_{3l}^{(j)} \right) \vec{i}_j \quad (4.10)$$

where  $a_1, a_2, a_3$  are the vector projections on the curvilinear axes.

Using formulas (4.8) and (4.10) from the equation (4.7) of the vector the following three scalar equations are obtained:

$$\begin{aligned} \rho \left[ \frac{\partial v_j}{\partial t} + \frac{v_1}{H_1} \frac{\partial v_j}{\partial q_1} + \frac{v_2}{H_2} \frac{\partial v_j}{\partial q_2} + \frac{v_3}{H_3} \frac{\partial v_j}{\partial q_3} + \frac{v_1 v_{j+1}}{H_1 H_{j+1}} \frac{\partial H_j}{\partial q_{j+1}} + \frac{v_1 v_{j+2}}{H_1 H_{j+2}} \frac{\partial H_j}{\partial q_{j+2}} - \right. \\ \left. \frac{v_{j+1}^2 \partial H_{j+1}}{H_j H_{j+1}} \frac{\partial}{\partial q_j} - \frac{v_{j+2}^2 \partial H_{j+2}}{H_j H_{j+2}} \frac{\partial}{\partial q_j} \right] + \end{aligned}$$

$$\begin{aligned}
 & \frac{v_j}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_3 H_1) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] = \\
 & \rho F_j + \frac{\vec{\tau}_{1j}}{H_1 H_2 H_3} \frac{\partial}{\partial q_1} (H_2 H_3) + \frac{\vec{\tau}_{2j}}{H_1 H_2 H_3} \frac{\partial}{\partial q_2} (H_3 H_1) + \frac{\vec{\tau}_{3j}}{H_1 H_2 H_3} \frac{\partial}{\partial q_3} (H_1 H_2) + \\
 & \frac{1}{H_1} \left( \frac{\partial \vec{\tau}_{1j}}{\partial q_1} + \vec{\tau}_{11} B_{11}^{(j)} + \vec{\tau}_{12} B_{21}^{(j)} + \vec{\tau}_{13} B_{31}^{(j)} \right) + \\
 & \frac{1}{H_2} \left( \frac{\partial \vec{\tau}_{2j}}{\partial q_2} + \vec{\tau}_{21} B_{12}^{(j)} + \vec{\tau}_{22} B_{22}^{(j)} + \vec{\tau}_{23} B_{32}^{(j)} \right) + \\
 & \frac{1}{H_3} \left( \frac{\partial \vec{\tau}_{3j}}{\partial q_3} + \vec{\tau}_{31} B_{13}^{(j)} + \vec{\tau}_{32} B_{23}^{(j)} + \vec{\tau}_{33} B_{33}^{(j)} \right) \quad (j = 1, 2, 3) \quad (4.11)
 \end{aligned}$$

Let us find the expression for the coefficients  $B_{kl}^{(m)}$ . From (4.9)

$$B_{kl}^{(m)} = \frac{\partial \vec{i}_k}{\partial q_l} \cdot \vec{i}_m \quad (4.12)$$

If  $\vec{i}, \vec{j}, \vec{k}$ , are orthogonal Cartesian coordinate axes and if the connection between the Cartesian and curvilinear coordinates is given by the formulas

$$x = x(q_1, q_2, q_3); \quad y = y(q_1, q_2, q_3); \quad z = z(q_1, q_2, q_3) \quad (4.13)$$

Then evidently

$$\begin{aligned}
 \vec{i}_k &= \frac{1}{H_k} \left( \frac{\partial x}{\partial q_k} \vec{i} + \frac{\partial y}{\partial q_k} \vec{j} + \frac{\partial z}{\partial q_k} \vec{k} \right) \\
 \vec{i}_m &= \frac{1}{H_m} \left( \frac{\partial x}{\partial q_m} \vec{i} + \frac{\partial y}{\partial q_m} \vec{j} + \frac{\partial z}{\partial q_m} \vec{k} \right) \quad (4.14)
 \end{aligned}$$

$$\frac{\partial \vec{i}_k}{\partial q_l} = \frac{\partial}{\partial q_l} \left( \frac{1}{H_k} \frac{\partial x}{\partial q_k} \right) \vec{i} + \frac{\partial}{\partial q_l} \left( \frac{1}{H_k} \frac{\partial y}{\partial q_k} \right) \vec{j} + \frac{\partial}{\partial q_l} \left( \frac{1}{H_k} \frac{\partial z}{\partial q_k} \right) \vec{k}$$

Therefore

$$B_k^{(m)} = \frac{1}{H_m} \left[ \frac{\partial x}{\partial q_m} \frac{\partial}{\partial q_l} \left( \frac{1}{H_k} \frac{\partial x}{\partial q_k} \right) + \frac{\partial y}{\partial q_m} \frac{\partial}{\partial q_l} \left( \frac{1}{H_k} \frac{\partial y}{\partial q_k} \right) + \frac{\partial z}{\partial q_m} \frac{\partial}{\partial q_l} \left( \frac{1}{H_k} \frac{\partial z}{\partial q_k} \right) \right] \quad (4.15)$$

Equation (4.11) with the presence of (4.15) are the required equations describing in differential form the momentum law.

Let us derive the energy equation: For this use the energy conservation law. Put

$$\Phi = U^*; \quad A = \frac{1}{2} \rho v^2 + \rho E; \quad B = \rho \vec{F} \cdot \vec{v} + \epsilon \quad (4.16)$$

$$C_1 = \vec{\tau}_1 \cdot \vec{v} + t_1; \quad C_2 = \vec{\tau}_2 \cdot \vec{v} + t_2; \quad C_3 = \vec{\tau}_3 \cdot \vec{v} + t_3$$

where  $U^*$  is the total energy,  $E$  is the internal energy of a unit mass,  $\rho \vec{F} \cdot \vec{v}$  is the force developed by the volume force  $\vec{\tau}_1 \cdot \vec{v}$ ,  $\vec{\tau}_2 \cdot \vec{v}$ ,  $\vec{\tau}_3 \cdot \vec{v}$  is the force developed by the surface force produced on the unit element of area perpendicular to the coordinate axes,  $\epsilon$  is the volume velocity excluding chemical light etc. energy,  $t_1$ ,  $t_2$ ,  $t_3$  is the heat flow across an area perpendicular to the axes.

Putting (4.16) in (3.14),

$$\begin{aligned} & \frac{d}{dt} \left[ \rho \left( \frac{v^2}{2} + E \right) \right] + \rho \left( \frac{v^2}{2} + E \right) \operatorname{div} \vec{v} = \rho \vec{F} \cdot \vec{v} + \epsilon + \\ & \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (\vec{\tau}_1 \cdot \vec{v} H_2 H_3) + \frac{\partial}{\partial q_2} (\vec{\tau}_2 \cdot \vec{v} H_3 H_1) + \frac{\partial}{\partial q_3} (\vec{\tau}_3 \cdot \vec{v} H_1 H_2) \right] + \\ & \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (t_1 H_2 H_3) + \frac{\partial}{\partial q_2} (t_2 H_3 H_1) + \frac{\partial}{\partial q_3} (t_3 H_1 H_2) \right] \end{aligned} \quad (4.17)$$

or

$$\begin{aligned} & \rho \frac{dE}{dt} + \left( E + \frac{v^2}{2} \right) \left( \frac{d\rho}{dt} + \rho \operatorname{div} \vec{v} \right) + \rho \vec{v} \cdot \frac{d\vec{v}}{dt} = \rho \vec{F} \cdot \vec{v} + \epsilon + \\ & \frac{\vec{v}}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (\vec{\tau}_1 H_2 H_3) + \frac{\partial}{\partial q_2} (\vec{\tau}_2 H_3 H_1) + \frac{\partial}{\partial q_3} (\vec{\tau}_3 H_1 H_2) \right] + \end{aligned} \quad (4.18)$$

$$\frac{\tau_1}{H_1} \frac{\partial \vec{v}}{\partial q_1} + \frac{\tau_2}{H_2} \frac{\partial \vec{v}}{\partial q_2} + \frac{\tau_3}{H_3} \frac{\partial \vec{v}}{\partial q_3} + \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (t_1 H_2 H_3) + \frac{\partial}{\partial q_2} (t_2 H_3 H_1) + \frac{\partial}{\partial q_3} (t_3 H_1 H_2) \right]$$

Using equations (4.2) and (4.6) for the scalar multiplication by  $\vec{v}$ , it is possible to produce evident simplification in the preceding equation.

Then there is obtained

$$\begin{aligned} \rho \frac{\partial E}{\partial t} + \left( E - \frac{v^2}{2} \right) \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_3 H_1) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] = \\ \epsilon + \frac{\tau_1}{H_1} \frac{\partial \vec{v}}{\partial q_1} + \frac{\tau_2}{H_2} \frac{\partial \vec{v}}{\partial q_2} + \frac{\tau_3}{H_3} \frac{\partial \vec{v}}{\partial q_3} + \\ \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (t_1 H_2 H_3) + \frac{\partial}{\partial q_2} (t_2 H_3 H_1) + \frac{\partial}{\partial q_3} (t_3 H_1 H_2) \right] \end{aligned} \quad (4.19)$$

This is the desired energy equation. With its writing in expanded form, have in view that the product of the vector velocity by the coordinates must be calculated with the help of (4.10).

In Cartesian coordinates  $x, y$  and  $z$  when  $H_1=H_2=H_3=1$  and  $B_{kl}^{(m)}=0$ , all the separate equations are essentially simplified and the following simple equations are obtained from (4.2), (4.11) and (4.18):

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \rho \operatorname{div} \vec{v} &= \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \\ \rho \frac{dv_x}{dt} + v_x \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \right) &= \rho F_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\ \rho \frac{dv_y}{dt} + v_y \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \right) &= \rho F_y + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\ \rho \frac{dv_z}{dt} + v_z \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \right) &= \rho F_z + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \\ \rho \frac{dE}{dt} + \left( E - \frac{v^2}{2} \right) \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \right) &= \\ \epsilon + \frac{\partial t_x}{\partial x} + \frac{\partial t_y}{\partial y} + \frac{\partial t_z}{\partial z} + \vec{\tau}_x \cdot \frac{\partial \vec{v}}{\partial x} + \vec{\tau}_y \cdot \frac{\partial \vec{v}}{\partial y} + \vec{\tau}_z \cdot \frac{\partial \vec{v}}{\partial z} \end{aligned} \quad (4.20)$$

The quantities  $Q_1, \vec{t}_1$  and  $t$ , which enter in the obtained equations are defined in the next paragraph.

#### 5. Expressions for Mass Flow, Momentum and Heat.

With the definitions of the mass flow<sup>1</sup>  $Q_1$ , Momentum  $\tau_{1k}$  and heat  $t_1$  across an area moving with the velocity  $\vec{v}$  of the gas perpendicular to the  $q_1$  axes, there will be considered that these quantities must be linear functions of the first derivatives of the hydrodynamic elements with respect to the Cartesian coordinates.<sup>2</sup> The coefficients of these linear functions must only depend on these hydrodynamic elements.

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<sup>1</sup>To simplify the explanation there is omitted the specification that the flow across a moving area is being discussed.

<sup>2</sup>This assumption may be applied, arising from numerous experiments dedicated to the study of different types of particular cases of transfer phenomena of mass, momentum and heat energy in gases. The results of these experiments showed that, observing certain conditions, the flow of these quantities actually appear linear functions of the first derivatives of the hydrodynamic elements  $v_x, v_y, v_z, \rho$  and  $T$  with respect to the Cartesian coordinates  $x, y, z$ .

It is clear that applying this assumption we narrow somewhat the class of gaseous motion which is easily studied with the aid of the obtained equations.

Actually, for this assumption to be correct, for example, it is necessary that the hydrodynamic elements be sufficiently accurately assumed linear approximations of a distance of the order of the mean free path of a molecule not defined by assigning its hydrodynamic elements and their first derivatives in some point of the volume. Not having data on the gas state in the volume, we formulate the flow, it would be impossible even to set the problem of finding expressions for the flows by the hydrodynamic elements and their first derivatives. However, in spite of some narrowing of the class of accessible by such considerations motions of a gas, tests and kinetic theory of gases say that the above-formulated basic assumption will with sufficient accuracy be fulfilled in a very broad class of gas motions which are of practical interest, or for this assumption to be correct it is sufficient the basic assumptions which are formulated in the present work.

Here the expression for the original flow is defined in Cartesian coordinates and then in general curvilinear orthogonal coordinates.

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Evidently they may not depend on the projections  $v_x$ ,  $v_y$ , and  $v_z$  and the velocity  $\vec{v}$ . Consequently they must be functions only of the density and temperature.<sup>1</sup>

Let us consider the desired linear forms for  $Q_x$ ,  $Q_y$ , and  $Q_z$ , starting with  $Q_x$ , put

$$\begin{aligned} Q_x = & A_x + a_{1x} \frac{\partial \rho}{\partial x} + a_{2x} \frac{\partial \rho}{\partial y} + a_{3x} \frac{\partial \rho}{\partial z} + \\ & b_{1x} \frac{\partial T}{\partial x} + b_{2x} \frac{\partial T}{\partial y} + b_{3x} \frac{\partial T}{\partial z} + \\ & c_{1x} \frac{\partial v_x}{\partial x} + c_{2x} \frac{\partial v_x}{\partial y} + c_{3x} \frac{\partial v_x}{\partial z} + \\ & d_{1x} \frac{\partial v_y}{\partial x} + d_{2x} \frac{\partial v_y}{\partial y} + d_{3x} \frac{\partial v_y}{\partial z} + \\ & e_{1x} \frac{\partial v_z}{\partial x} + e_{2x} \frac{\partial v_z}{\partial y} + e_{3x} \frac{\partial v_z}{\partial z} \end{aligned} \quad (5.1)$$

Evidently with the variation of the direction of the x-axis in the opposite,  $Q_x$  must change in sign. Consequently in the right side of formula (5.1) there must be neither terms nor variations of sign with the variation of the x-axis to the opposite. Hence

$$A_x = a_{2x} = a_{3x} = b_{2x} = b_{3x} = c_{1x} = d_{2x} = d_{3x} = e_{2x} = e_{3x} = 0 \quad (5.2)$$

Further,  $Q_x$  must not depend on the directions of the y and z axes. Consequently

$$c_{2x} = c_{3x} = d_{1x} = e_{1x} = 0 \quad (5.3)$$

Therefore

$$Q_x = a_{1x} \frac{\partial \rho}{\partial x} + b_{1x} \frac{\partial T}{\partial x} \quad (5.4)$$

---

<sup>1</sup>Being discussed are flows across areas moving with the gas. The relative velocity of the gas with respect to the area is always zero. Consequently the values of  $v_x$ ,  $v_y$ , and  $v_z$  are not reflected on the flows.

---

Since the  $x$ ,  $y$  and  $z$  axes are completely equal then

$$a_{1x} = a_{2y} = a_{3z} = D_1; \quad b_{1x} = b_{2y} = b_{3z} = D_2 \quad (5.5)$$

Similarly

$$Q_x = D_1 \frac{\partial \rho}{\partial x} + D_2 \frac{\partial T}{\partial x}, \quad Q_y = D_1 \frac{\partial \rho}{\partial y} + D_2 \frac{\partial T}{\partial y}, \quad Q_z = D_1 \frac{\partial \rho}{\partial z} + D_2 \frac{\partial T}{\partial z} \quad (5.6)$$

The coefficient  $D_1$  is called the coefficient of self-diffusion of density, and  $D_2$  is called the coefficient of self-diffusion of temperature.

For linear forms having the heat flow  $t_x$ ,  $t_y$ ,  $t_z$  repeating that only the statement of the considerations lead to

$$t_x = K_1 \frac{\partial \rho}{\partial x} + K_2 \frac{\partial T}{\partial x}; \quad t_y = K_1 \frac{\partial \rho}{\partial y} + K_2 \frac{\partial T}{\partial y}; \quad t_z = K_1 \frac{\partial \rho}{\partial z} + K_2 \frac{\partial T}{\partial z} \quad (5.7)$$

where  $K_1$  is the coefficient of heat conduction by density and  $K_2$  is the coefficient of heat conduction by temperature. The coefficients  $D_1$  and  $D_2$  and  $K_1$  and  $K_2$  are encountered in the kinetic theory of gases.

Finally, let us establish for Cartesian coordinates the aspect of linear form giving the projections of the vectors  $\vec{\tau}_x$ ,  $\vec{\tau}_y$ , and  $\vec{\tau}_z$ , on the  $x$ ,  $y$ ,  $z$  axes.

Turning to equation (3.14) we write it in Cartesian coordinates applicable to the momentum law. Put

$$\begin{aligned} \vec{\Phi} &= \vec{L}, \quad \vec{A} = \vec{r} \times \rho \vec{v}, \quad \vec{B} = \vec{r} \times \rho \vec{F} \\ \vec{C}_1 &= \vec{r} \times \vec{\tau}_x, \quad \vec{C}_2 = \vec{r} \times \vec{\tau}_y, \quad \vec{C}_3 = \vec{r} \times \vec{\tau}_z \end{aligned} \quad (5.8)$$

where  $\vec{L}$  is the momentum,  $\vec{r}$  is the radius-vector of a point of the moving medium,  $\vec{r} \times \rho \vec{F}$  is the mass force applied to unit volume,  $\vec{r} \times \vec{\tau}_x$ ,  $\vec{r} \times \vec{\tau}_y$ ,  $\vec{r} \times \vec{\tau}_z$  are the momentum flow across a moving area perpendicular to the axes.

Substituting (5.8) in (3.14)

$$\frac{d}{dt} (\vec{r} \times \rho \vec{v}) + (\vec{r} \times \rho \vec{v}) \operatorname{div} \vec{v} = \vec{r} \times \rho \vec{F} + \frac{\partial}{\partial x} (\vec{r} \times \vec{\tau}_x) + \frac{\partial}{\partial y} (\vec{r} \times \vec{\tau}_y) + \frac{\partial}{\partial z} (\vec{r} \times \vec{\tau}_z) \quad (5.9)$$

or

$$\begin{aligned} & \vec{r} \times \left[ \frac{d}{dt} (\rho \vec{v}) + \rho \vec{v} \operatorname{div} \vec{v} - \rho \vec{F} - \frac{\partial \tau_x}{\partial x} - \frac{\partial \tau_y}{\partial y} - \frac{\partial \tau_z}{\partial z} \right] + \\ & \frac{d\vec{r}}{dt} \times \rho \vec{v} + \frac{\partial \vec{r}}{\partial x} \times \vec{\tau}_x + \frac{\partial \vec{r}}{\partial y} \times \vec{\tau}_y + \frac{\partial \vec{r}}{\partial z} \times \vec{\tau}_z = 0 \end{aligned} \quad (5.10)$$

Here the square bracket, or the basis of (4.4), is zero.

Moreover,

$$\frac{d\vec{r}}{dt} \times \rho \vec{v} = \vec{v} \times \rho \vec{v} = 0, \quad \frac{\partial \vec{r}}{\partial x} = \vec{i}, \quad \frac{\partial \vec{r}}{\partial y} = \vec{j}, \quad \frac{\partial \vec{r}}{\partial z} = \vec{k} \quad (5.11)$$

Therefore

$$\vec{i} \times \vec{\tau}_x + \vec{j} \times \vec{\tau}_y + \vec{k} \times \vec{\tau}_z = 0 \quad (5.12)$$

This vector equality is equivalent to three scalar equalities

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{zx} = \tau_{xz}$$

representing the well-known three-dimensional property of the symmetric pressure tensor on a continuous medium with self-diffusion.

Keeping this in mind and corresponding to the general considerations, we obtain

$$\begin{aligned} \tau_{ik} = & A_{ik} + a_{ik}^{(1)} \frac{\partial \rho}{\partial x} + a_{ik}^{(2)} \frac{\partial \rho}{\partial y} + a_{ik}^{(3)} \frac{\partial \rho}{\partial z} + \\ & b_{ik}^{(1)} \frac{\partial T}{\partial x} + b_{ik}^{(2)} \frac{\partial T}{\partial y} + b_{ik}^{(3)} \frac{\partial T}{\partial z} + \\ & c_{ik}^{(1)} \frac{\partial v_x}{\partial x} + c_{ik}^{(2)} \frac{\partial v_y}{\partial y} + c_{ik}^{(3)} \frac{\partial v_z}{\partial z} + \end{aligned} \quad (5.13)$$

$$\begin{aligned} & d_{ik}^{(1)} \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) + d_{ik}^{(2)} \left( \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) + d_{ik}^{(3)} \left( \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) + \\ & e_{ik}^{(1)} \left( \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) + e_{ik}^{(2)} \left( \frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right) + e_{ik}^{(3)} \left( \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) \end{aligned}$$



The quantities  $A_{ik}$ , as the other coefficients in (5.13), do not depend on the values of the derivatives of the hydrodynamic elements with respect to the coordinates. They may be found by those values which they assume in a gas with constant hydrodynamic elements

$$A_k = -p, \quad A_{ik} = 0, \quad i \neq k \quad (5.14)$$

where  $p$  is the pressure in a gas with constant hydrodynamic elements which, for an ideal gas, is determined by Klappenrod's equation

$$p = R\rho T \quad (5.15)$$

Moreover, since the values of  $\tau_{ik}$  do not vary with variation of axial direction and back, and the derivatives of  $\rho$  and  $T$  with respect to the coordinates for this same transformation change sign conversely, then

$$a_{ik}^{(1)} = a_{ik}^{(2)} = a_{ik}^{(3)} = b_{ik}^{(1)} = b_{ik}^{(2)} = b_{ik}^{(3)} = 0 \quad (5.16)$$

Certainly, according to (5.12) it is evident that

$$e_{ik}^{(1)} = e_{ik}^{(2)} = e_{ik}^{(3)} = 0 \quad (5.17)$$

Consequently the  $\vec{\tau}_{ik}$  (components of the pressure tensor) influence by linear functions only the components of the deformation velocity tensor. As is known [5], for this the following equalities are sufficient

$$\begin{aligned} \tau_{xx} &= -p + \mu_1 \operatorname{div} \vec{v} + 2\mu \frac{\partial v_x}{\partial x}, & \tau_{xy} &= \mu \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \\ \tau_{yy} &= -p + \mu_1 \operatorname{div} \vec{v} + 2\mu \frac{\partial v_y}{\partial y}, & \tau_{yz} &= \mu \left( \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \\ \tau_{zz} &= -p + \mu_1 \operatorname{div} \vec{v} + 2\mu \frac{\partial v_z}{\partial z}, & \tau_{zx} &= \mu \left( \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \end{aligned} \quad (5.18)$$

where  $\mu_1$  and  $\mu$  are certain functions, generally speaking, of the density  $\rho$  and the temperature  $T$ . The quantity  $\mu$ , as is known, is called the

coefficient of viscosity and the quantity  $\mu_1$  - the coefficient of second viscosity.

If, in the reasoning these tensors are introduced: pressure  $\vec{T}$ , deformation velocity  $\vec{II}$  and unity  $\vec{I}$ , then all the preceding equalities unite into one:

$$\vec{T} = (-p + \mu_1 \operatorname{div} \vec{v}) \vec{I} + 2\mu \vec{II} \quad (5.19)$$

which is always convenient for transformation to any orthogonal curvilinear coordinates. Transforming (5.6) and (5.7) to curvilinear orthogonal coordinates we obtain

$$Q_i = D_1 \frac{1}{H_i} \frac{\partial \rho}{\partial q_i} + D_2 \frac{1}{H_i} \frac{\partial T}{\partial q_i}; \quad t_i = K_1 \frac{1}{H_i} \frac{\partial \rho}{\partial q_i} + K_2 \frac{1}{H_i} \frac{\partial T}{\partial q_i} \quad (5.20)$$

Moreover, transforming (5.19) to curvilinear coordinates executing the usual calculations [5], we obtain the last of the desired formulas:

$$\begin{aligned} \tau_{ii} = & -\rho + \mu_1 \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q_1} (v_1 H_2 H_3) + \frac{\partial}{\partial q_2} (v_2 H_3 H_1) + \frac{\partial}{\partial q_3} (v_3 H_1 H_2) \right] + \\ & 2\mu \left[ \frac{1}{H_i} \frac{\partial v_i}{\partial q_i} + \frac{v_{i+1}}{H_i H_{i+1}} \frac{\partial H_i}{\partial q_{i+1}} + \frac{v_{i+2}}{H_i H_{i+2}} \frac{\partial H_i}{\partial q_{i+2}} \right] \\ \tau_{ik} = & \mu \left[ \frac{1}{H_k} \frac{\partial v_i}{\partial q_k} + \frac{1}{H_i} \frac{\partial v_k}{\partial q_i} - \frac{v_i}{H_i H_k} \frac{\partial H_i}{\partial q_k} - \frac{v_k}{H_i H_k} \frac{\partial H_k}{\partial q_i} \right] \end{aligned} \quad (5.21)$$

The expressions obtained for mass flow, momentum and heat contain six coefficients:  $D_1, D_2, K_1, K_2, \mu_1, \mu$ .

#### 6. Expressions for the coefficients $\mu_1, D_1, D_2, K_1, K_2$ .

To establish expressions for the coefficients  $\mu_1, D_1, D_2, K_1$  and  $K_2$  dimensional theory is used.

If  $c_v$  the coefficient of specific heat for constant volume is introduced into the reasoning, expressed in mechanical not heat units,

then there will be the following relation between the dimensions:

$$[\mu_1] = [\mu], \quad [D_1] = \left[ \frac{\mu}{\rho} \right], \quad [D_2] = \left[ \frac{\mu}{T} \right], \quad [K_1] = \left[ \frac{\mu c_v T}{\rho} \right], \quad [K_2] = [\mu c_v] \quad (6.1)$$

Consequently, we may put

$$\mu_1 = a\mu, \quad D_1 = \frac{\mu}{\rho} \alpha_1, \quad D_2 = \frac{\mu}{T} \alpha_2, \quad K_1 = \frac{\mu c_v T}{\rho} \beta_1, \quad K_2 = \mu c_v \beta_2 \quad (6.2)$$

where  $a, \alpha_1, \alpha_2, \beta_1$  and  $\beta_2$ , are dimensionless functions of dimensionless parameters defining the state of the gas at equilibrium because the formulas for the flows with these same coefficients prove applicable for the whole considered gas states including those which are as close as one pleases to the equilibrium state.

The equilibrium state of the given ideal gas is completely defined by giving its density  $\rho$  and its temperature  $T$ . Out of these qualities it is impossible to establish one dimensionless combination. Consequently, for an ideal gas the quantities  $a, \alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  will be constants depending on the type of gas and consequently, the coefficients  $\mu_1, D_1, D_2, K_1$  and  $K_2$  may be considered known with the accuracy of the constants  $a, \alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  if the coefficient of viscosity  $\mu$  is known or the coefficient of specific heat for constant volume  $c_v$  is constant for an ideal gas.

## 7. The Coefficients $a, \alpha_1, \alpha_2, \beta_1$ and $\beta_2$ .

Kinetic theory of gases permits the expectation that the numerical coefficients  $a, \alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  will be quantities of the order of unity. Generally speaking, these quantities must be found in a corresponding way from the set-up tests.

In the present work we do not dispose of all these coefficients by test values but nonetheless relying on some test results and on

considerations of kinetic theory of gases we give the numerical value of these coefficients of a monatomic gas.

First of all from kinetic gas theory it is well known that for all monatomic gases [6]

$$a = -\frac{2}{3} \quad (7.1)$$

Moreover, from tests on self-diffusion of gases [7] it has been established that for all monatomic gases the coefficient (to the limits of test accuracy) is the same

$$\alpha_1 = 1.30 \quad (7.2)$$

which is close enough to the theoretical value of this coefficient obtained for different molecular model. Having  $\alpha_1$ , it is not difficult to establish at once the value of  $\beta_1$ .

By the same sense of the coefficient  $D_1$  it may be confirmed that for constant temperature  $T$  and variable density  $\rho$  across an element of area  $dS$  with normal  $\vec{n}$  and time  $dt$  because of the density of self-diffusion the mass transport will be  $\Delta m$ , yielding the formula

$$\Delta m = D_1 \frac{\partial \rho}{\partial n} dS dt \quad (7.3)$$

This mass possesses the heat energy  $\Delta q$ , where

$$\Delta q = \Delta m c_v T = c_v T D_1 \frac{\partial \rho}{\partial n} dS dt \quad (7.4)$$

This heat energy is the heat energy passed across our area in time  $dt$  as a consequence of the density of heat conduction. Therefore,

$$\Delta q = K_1 \frac{\partial \rho}{\partial n} dS dt \quad (7.5)$$

Equating (7.4) and (7.5), we obtain  $\alpha_1 = \beta_1$ . Therefore for monatomic gases

$$\beta_1 = 1.30 \quad (7.6)$$

Moreover turning to the establishment of heat transfer across a plane gas layer enclosed between two walls in a distance  $\Delta l$  for difference in wall temperatures  $\Delta T$ . If heat flow be denoted by  $q$ , then from experimental data it is easy to find the value of  $f$  determined by the formula

$$f = \frac{q}{\mu c_v \Delta T / \Delta l} \quad (7.7)$$

The average test value of this quantity for a monatomic gas [8] equals 2.51. On the other hand, it will be proved below (Sections 9, 10) that

$$f = \beta_2 - \beta_1 \quad (7.8)$$

Therefore, for monatomic gases

$$\beta_2 = 3.81 \quad (7.9)$$

Let us now find  $\alpha_2$ .

Let us consider the heat transfer in a still gas of constant density and variable temperature. We denote by  $\Delta q$  the amount of heat energy passing across an element of area  $dS$  with normal  $\vec{n}$  in the time  $dt$ . Evidently

$$\Delta q = K_2 \frac{\partial T}{\partial n} dS dt \quad (7.10)$$

The quantity  $\Delta q$  consists of two components:  $\Delta q_1$  and  $\Delta q_2$ .

The first component,  $\Delta q_1$ , represents the heat energy flowing across an element of area taking into account only the variation of temperature, neglecting the heat of self-diffusion.

This quantity may be calculated by means of the theoretical formula

$$\Delta q_1 = f \mu c_v \frac{\partial T}{\partial n} dS dt \quad (7.11)$$

As a basis for certain applications of this formula to monatomic gases this circumstance serves: that for different molecular models of

monatomic gases there is always obtained for the numerical coefficient  $f^*$  an approximate value [9]. Namely, for all models of monatomic molecules considered theoretically,  $f^*$  is included between the limits 2.50-2.52. Consequently we use

$$\Delta q_1 = 2.51 \mu c_v \frac{\partial T}{\partial n} dS dt \quad (7.12)$$

The second component  $\Delta q_2$  represents the amount of heat energy transported by the heat of self-diffusion of mass flow. If  $\Delta m$  denotes the mass penetrating an element of area as a consequence of the heat of self-diffusion, then

$$\Delta m = D_2 \frac{\partial T}{\partial n} dS dt \quad (7.13)$$

This mass possesses the heat energy  $\Delta q_2$ , where

$$\Delta q_2 = \Delta m c_v T = c_v T D_2 \frac{\partial T}{\partial n} dS dt \quad (7.14)$$

Equating the two expressions obtained for  $\Delta q$

$$\alpha_2 = \beta_2 - f^* \quad (7.15)$$

Therefore,

$$\alpha_2 = 1.30 \quad (7.16)$$

Corresponding to the preliminary expectations, all the coefficients  $\alpha_1, \alpha_2, \beta_1, \beta_2$  and  $a$  are terms of the order of unity.

Remarks: The relative magnitude of the different terms in the equations of motion under different conditions will be different. It is possible to specify such conditions of gas motion when the fundamental values will have self-diffusion terms; it is possible to specify such conditions of motion when the fundamental value has only terms related to the pressure tensor, etc. Consequently, it is impossible to speak of the relative magnitude of different terms in the motion equations not isolated from a definite class of motions.

From the practical point of view an important class of motions is that motion when all the hydrodynamic elements (velocity  $v$ , density  $\rho$  and temperature  $T$ ) vary from magnitude to magnitude by the order of their distance from the order of the same length.

To such motions belongs, for example, motion in the boundary layer at high speeds. In these motions at points separated by a distance of the order of the boundary-layer thickness  $\delta$ , generally speaking, the hydrodynamic elements differ by the magnitude of the order of these same hydrodynamic elements.

If that motion with one characteristic distance for all hydrodynamic elements is kept in mind then the standard transformation to dimensionless quantities leads at once to the conclusion that, in these motions, all self-diffusion, heat conduction and viscosity terms have the same relative magnitude if only the constants  $a$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$ , and  $\beta_2$  have one order.

Since, according to kinetic gas theory, all these constants are of the order of unity, then it follows that for the motion considered if only one term related to transport phenomena is retained in the equations, then all the other terms related to these phenomena must be retained.

Moreover, this means that for the motion of the considered class either completely exclusive equations or the equations of motion of an ideal compressible fluid may be used.

#### 8. Boundary Conditions for the System of Differential Equations of Gas Motion

To integrate the system of differential equations of gas motion which is obtained, boundary conditions are necessary which occur on the

surfaces of rigid bodies of gas streamlines and boundary conditions at infinity in that case, when the domain occupied by the gas extends to infinity.<sup>1</sup>

The question of the boundary conditions at infinity are always easily solved. Evidently these conditions must include the density, temperature, and velocity components assigned at infinity.

The question of the boundary conditions on the surface of a streamlined rigid body is more complex to solve.

First of all, it is completely clear that through the surface of a streamlined rigid body the mass of gas does not penetrate.

If the normal to the surface  $S$  of a rigid body be denoted by  $\vec{n}$  and it is assumed that the body streamlines do not move in space then this physical fact, evidently, is described in the following way:

$$\left( D_1 \frac{\partial \rho}{\partial n} + D_2 \frac{\partial T}{\partial n} \right)_{\vec{s}} = (\rho v_n) S \quad (8.1)$$

In order to obtain the boundary conditions on the surface of the streamlined rigid body, we assume that immediately on the surface the gas moves either very slowly or at rest.<sup>2</sup>

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<sup>1</sup> Since the order of the obtained system at unity is higher than the order of that system which occurs for incomplete consideration of transfer phenomena, then the old boundary conditions do not correspond to the setup of the problem if only because its number doesn't correspond to the new order of the system of the motion equations.

<sup>2</sup> It is possible to reduce certain physical occurrences on the basis of this assumption. Visually the smooth streamlined surface may be assumed for a gas of micro-motion close to the surface, the gas will be found in the conditions, as close to the streaming conditions across a very slightly porous medium.

Well-known are the enormous coefficients of the resistance of gas motion across different kinds of slightly latticed and porous media and also known the extremely inconsiderable input of a gas for motion through a latticed and porous medium which for sufficient smallness of pores is practically zero.

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Neglecting small velocities of the gas motion directed tangent to the surface  $S$ , and denoting by  $\vec{l}_1$ , and  $\vec{l}_2$  the vectors tangent to the surface  $S$ , we obtain two boundary conditions

$$\left( v_{l_1} \right)_S = 0 \quad \left( v_{l_2} \right)_S = 0 \quad (8.2)$$

In the gas at rest we have a static pressure distribution. Consequently, with the known approximation, it is possible to assume

$$\left( \frac{\partial \rho}{\partial n} \right)_S = (\rho F_n)_S \quad (8.3)$$

where  $F_n$  is the projection of the mass force on the normal to the surface.

The boundary condition (8.2) is confirmed well enough by experiment. Boundary condition (8.3) must be verified by tests and at present may be considered as a probable hypothesis.

#### 9. Example of the Integration of the System of Equations of Gas Motion

Let us consider the uniform simple problem of heat transfer through a gas layer between two parallel planes for the conditions that the difference in the temperature of the walls is small and there is no mass force.

Let the two parallel planes be  $x = -l$  and  $x = +l$ . Let the temperature of these planes equal, respectively,  $T_1$  and  $T_2$ . Let the gas between the planes be in such a quantity that it would have density  $\rho_0$  if it were constant throughout the space occupied by the gas. Finally, let the quantity

$$\epsilon^* = \frac{T_2 - T_1}{T_2 + T_1} \quad (9.1)$$

be so small that its squares may be neglected in comparison to one.

The considered problem corresponds to the following transformation of the system of equations of gas motion:

$$\begin{aligned}\frac{d}{dx}(\rho v) &= \frac{d}{dx}\left(D_1 \frac{d\rho}{dx} + D_2 \frac{dT}{dx}\right) \\ \frac{d}{dx}(\rho v^2) &= \frac{d}{dx}\left(-R\rho T + \frac{4}{3}\mu \frac{dv}{dx}\right) \\ \rho c_v v \frac{dT}{dx} + \left(c_v T - \frac{v^2}{2}\right) \frac{d}{dx}(\rho v) &= \frac{d}{dx}\left(K_1 \frac{d\rho}{dx} + K_2 \frac{dT}{dx}\right) + \\ &\quad \left(-R\rho T + \frac{4}{3}\mu \frac{dv}{dx}\right) \frac{dv}{dx}\end{aligned}\quad (9.2)$$

For simplicity, the gas is considered monatomic here; this does not affect the generality. To determine the constants of integration, we have the following conditions:

$$\rho v = D_1 \frac{d\rho}{dx} + D_2 \frac{dT}{dx}, \quad T=T_2, \quad T \frac{d\rho}{dx} + \rho \frac{dT}{dx} = 0 \quad \text{for } x = +l \quad (9.3)$$

$$\rho v = D_1 \frac{d\rho}{dx} + D_2 \frac{dT}{dx}, \quad T=T_1, \quad T \frac{d\rho}{dx} + \rho \frac{dT}{dx} = 0 \quad \text{for } x = -l \quad (9.4)$$

$$\frac{1}{2l} \int_{-l}^{+l} \rho \, dx = \rho_0 \quad (9.5)$$

At first glance it may appear that we have seven conditions to define six constants of integration.

However, consideration of the first of equations (9.2) shows at once that the first of conditions (9.3) and (9.4) are not independent and follow from each other by virtue of this differential equation.

Because the quantity  $\epsilon^*$  is assumed small, essentially (9.2) and conditions (9.3) and (9.5) are linearized. Put

$$\rho = \rho_0 + \rho', \quad T = \frac{1}{2}(T_2 + T_1) + T' = T_0 + T', \quad v = v' \quad (9.6)$$

$$D_i = D_i^{(0)} + D_i', \quad K_i = K_i^{(0)} + K_i', \quad \mu = \mu_0 + \mu'$$

where the primed quantities are small.

Then to determine  $\rho'$ ,  $v'$  and  $T'$  we obtain the following system of equations:

$$\rho_0 \frac{dv'}{dx} = D_1(o) \frac{\partial^2 \rho'}{\partial x^2} + D_2(o) \frac{d^2 T'}{dx^2}$$

$$\frac{4}{3} \mu_0 \frac{d^2 v'}{dx^2} - R \left( T_0 \frac{d\rho'}{dx} + \rho_0 \frac{dT'}{dx} \right) = 0 \quad (9.7)$$

$$K_1(o) \frac{d^2 \rho'}{dx^2} + K_2(o) \frac{d^2 T'}{dx^2} = c_p \rho_0 T_0 \frac{dv'}{dx}$$

$$R = c_p - c_v \quad (9.8)$$

Moreover, we also have the following system

$$\rho_0 v' = D_1(o) \frac{d\rho'}{dx} + D_2(o) \frac{dT'}{dx} \quad \text{for } x = +l$$

$$T' = -\frac{T_2 - T_1}{2}, \quad T_0 \frac{d\rho'}{dx} + \rho_0 \frac{dT'}{dx} = 0 \quad \text{for } x = +l$$

$$T' = -\frac{T_2 - T_1}{2}, \quad T_0 \frac{d\rho'}{dx} + \rho_0 \frac{dT'}{dx} = 0 \quad \text{for } x = -l \quad (9.9)$$

$$\int_{-l}^{+l} \rho' dx = 0$$

Integrating the first and third of equations (9.7) and taking into account the first of conditions (9.9) we obtain

$$D_1(o) \frac{d\rho'}{dx} + D_2(o) \frac{dT'}{dx} = \rho_0 v'$$

$$K_1(o) \frac{d\rho'}{dx} + K_2(o) \frac{dT'}{dx} = c_p \rho_0 T_0 v' + A \mu_0 c_v T_0 \quad (9.10)$$

where  $A \mu_0 c_v T_0$  is the constant of integration. Because, by virtue of (6.2),

$$D_1(o) = \frac{\mu_0}{\rho_0} \alpha_1, \quad K_1(o) = \mu_0 c_v \beta_1 \frac{T_0}{\rho_0}$$

$$D_2(o) = \frac{\mu_0}{T_0} \alpha_2, \quad K_2(o) = \mu_0 c_v \beta_2 \quad (9.11)$$

Then solving (9.10) for  $\rho'/dx$  and  $dT'/dx$ , we obtain

$$\begin{aligned}\frac{\partial \rho'}{\partial x} &= \frac{\rho_0}{\Delta} \left[ \left( \beta_2 - \frac{c_p}{c_v} \alpha_2 \right) \frac{\rho_0}{\mu_0} v' - \alpha_2 A \right] \\ \frac{dT'}{dx} &= \frac{T_0}{\Delta} \left[ - \left( \beta_1 - \frac{c_p}{c_v} \alpha_1 \right) \frac{\rho_0}{\mu_0} v' + \alpha_1 A \right]\end{aligned}$$

where  $\left( \Delta = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \right)$  (9.12)

Putting (9.12) into the second of equations (9.7),

$$v^2 \frac{d^2 v'}{dx^2} - v' = -uA \quad (9.13)$$

In this equation

$$\begin{aligned}v^2 &= \frac{4}{3} \frac{-\Delta}{(c_p/c_v)(\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)} \frac{\mu_0^2}{\rho_0 \rho_0} \\ u &= \frac{\alpha_2 - \alpha_1}{(c_p/c_v)(\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)} \frac{\mu_0}{\rho_0}\end{aligned}$$

where  $(p_0 = R \rho_0 T_0)$  (9.14)

Equation (9.13) is easily integrated. After integration we obtain

$$v' = uA + B_1 e^{x/\nu} + B_2 e^{-x/\nu} \quad (9.15)$$

where  $B_1$  and  $B_2$  are integration constants.

With the aid of (9.15) it is easy to find  $\rho'$  and  $T'$  from (9.12).

Making the calculations, we obtain

$$\begin{aligned}\rho' &= \frac{\rho_0 A}{(c_p/c_v)(\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)} x + \\ &\quad \frac{\rho_0}{\Delta} \left( \beta_2 - \frac{c_p}{c_v} \alpha_2 \right) v \frac{\rho_0}{\mu_0} (B_1 e^{x/\nu} - B_2 e^{-x/\nu}) + C_1 \\ T' &= \frac{-T_0 A}{(c_p/c_v)(\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)} x - \\ &\quad \frac{T_0}{\Delta} \left( \beta_1 - \frac{c_p}{c_v} \alpha_1 \right) v \frac{\rho_0}{\mu_0} (B_1 e^{x/\nu} - B_2 e^{-x/\nu}) + C_2\end{aligned} \quad (9.16)$$

where  $C_1$  and  $C_2$  are integration constants.

The constants of integration are  $A, B_1, B_2, C_1, C_2$  are determined from the remaining unused conditions, thus

$$v' = \frac{T_2 - T_1}{2l} \frac{\mu_0}{\rho_0 T_0} (\alpha_2 - \alpha_1), \quad \rho' = -\frac{T_2 - T_1}{T_2 + T_1} \rho_0 \left( \frac{x}{l} \right), \quad T' = \frac{T_2 - T_1}{2} \left( \frac{x}{l} \right) \quad (9.17)$$

From these formulas it is not difficult to find now the heat flow  $t_x$  across any element of area perpendicular to the  $x$  axis. Applying the linearized formula (5.7) and using the formulas (6.2) we obtain

$$t_x = \mu_0 c_v (\beta_2 - \beta_1) \frac{T_2 - T_1}{2l} \quad (9.18)$$

From formulas (9.17) and (9.18) certain conclusions may be made.

First, if the value  $v'$  can be found from experiment, then from the first of the formulas (9.17) we have a method of determining  $(\alpha_2 - \alpha_1)$ .

Second, from formula (9.18) there immediately results that the variation of heat flow  $t_x$  permits the determination of  $(\beta_2 - \beta_1)$ .

Finally, third, from formulas (6.23) there results that all experimental work devoted to finding the quantity  $\beta_2$  by means of the formula

$$\beta_2 = \frac{2lt_x}{(T_2 - T_1)\mu_0 c_v} \quad (9.19)$$

which is used only in experimental work, simultaneously yielded the value of  $(\beta_2 - \beta_1)$ .

# 10. Second Example of Integration of the System of Equations of Gas Motion

Let us now consider the uniform problem of the steady heat transfer across a gas layer between two parallel planes with the conditions that there is no mass force, but the temperature difference at the walls is not small:

In conformance with the kinetic theory of gases, we will consider that the viscosity  $\mu$  depends only on the temperature and is a known function of the temperature. In this case we must again integrate the system (9.2) with the conditions (9.3) - (9.5) to determine the integration constants.

On the basis of the first of formulas (9.17) which gives the order of the gas velocity  $v$  in the case considered, it is possible to confirm that in a very broad class of cases the velocity will be a small quantity and with large jump of temperature.

Using this in the second and third of equations (9.2), we neglect terms having order  $\mu^2$  and  $\mu^3$ . Then the system (9.2) becomes

$$\frac{d}{dx}(\rho v) = \frac{d}{dx} \left( D_1 \frac{dp}{dx} + D_2 \frac{dT}{dx} \right) \quad (10.1)$$

$$\frac{d}{dx} (R\rho T) = 0$$

$$C_v \frac{d}{dx} (\rho v T) = \frac{d}{dx} \left( K_1 \frac{dp}{dx} + K_2 \frac{dT}{dx} \right) - R\rho T \frac{dv}{dx}$$

The first of equations (10.1) immediately integrates and after satisfaction of the first boundary condition in (9.3), and therefore the first boundary condition (9.4) yields

$$\rho v = D_1 \frac{dp}{dx} + D_2 \frac{dT}{dx} \quad (10.2)$$

The second of equations (10.1) also integrates and yields

$$\rho RT = C_1 \quad (10.3)$$

where  $C_1$  is an integration constant.

The integral (10.3) guarantees fulfillment of the third condition of (9.3) and the third condition of (9.4).

Moreover if (10.3) is taken into account then the third of equations (10.1) integrates and yields

$$c_v \rho v T = K_1 \frac{d\rho}{dx} + K_2 \frac{dT}{dx} - C_1 v = C_2 \quad (10.4)$$

where  $C_2$  is an integration constant.

Put the velocity  $v$  from (10.2) into (10.4). Expression (6.2) for the coefficients  $D_1$ ,  $D_2$ ,  $K_1$  and  $K_2$  is used and substituted in (10.4) and, finally, with the aid of (10.3) the density  $\rho$  is eliminated from (10.4). Then we obtain the very simple equation

$$\mu \frac{dT}{dx} = \frac{C_2}{c_v(\beta_2 - \beta_1) - c_p(\alpha_2 - \alpha_1)} \quad (10.5)$$

Integrating this equation and satisfying the second of the boundary conditions (9.3) and (9.4) we obtain

$$\left[ \int_{T_1}^{T_2} \mu(T) dT \right]^{-1} \left[ \int_{T_1}^T \mu(T) dT - \int_T^{T_2} \mu(T) dT \right] = \frac{x}{l} \quad (10.6)$$

The right side of (10.6) is a known function of temperature and consequently, from (10.6), it is possible to find the temperature distribution independent of  $x$ .

Using (10.3), condition (9.5) and transforming from the integration variable  $x$  to the new variable  $T$  with the aid of (10.5), we find  $C_1$  and obtain a formula for the density

$$\rho = \frac{\rho_0}{T} \int_{T_1}^{T_2} \mu(T) dT \left[ \int_{T_1}^{T_2} \frac{1}{T} \mu(T) dT \right]^{-1} \quad (10.7)$$

Moreover, substituting the expression for  $T$  and  $\rho$  in (10.2), using (10.5) and the expression for  $C_2$ , we find  $v$ . Thus

$$v = \frac{\alpha_2 - \alpha_1}{2\lambda\rho} \int_{T_1}^{T_2} \frac{1}{T} \mu(T) dT \quad (10.8)$$

Formulas (10.6), (10.7), and (10.8) solve the problem posed.

Finally, without difficulty, it is possible to obtain an expression for the heat flow  $t_x$ . As the velocity, it is a constant and given by the formula:

$$t_x = C_v(\beta_2 - \beta_1) \frac{1}{2l} \int_{T_1}^{T_2} \mu(T) dT \quad (10.9)$$

Let us make some observations on the formulas obtained.

First of all, let us note the circumstance that we may not find the different quantities  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$ , and  $\beta_2$  from these formulas but only their difference. Further, let us remark that formulas (10.6) and (10.7) permit us to find easily the dependence of viscosity on temperature with the aid of the measured density or temperature.



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